SYNTHESIS OF A PRESCRIBED TEMPERATURE–TIME PROCESS BY MEANS OF SEMICONDUCTOR-TYPE THERMOELECTRIC CELLS

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The problem of programming a temperature–time process at the active surface of a thermoelectric battery is solved by synthesis of a control function. The equation of the controlling current–time curve is found. Considered are the problem of implementing a monotonic temperature decrease and the problem of generating complex temperature waveforms.

Measurement procedures as well as biological and physical experiments may require that the temperature of the object be varied according to a certain temperature–time relation. Whenever cooling of the object is required within some time interval, one may use semiconductor-type thermoelectric batteries for that purpose. The temperature at the battery surface can then be controlled by an appropriate variation of the supply current with time.

Synthesizing a programmed control of a thermoelectric cooling or heating process means determining the current–time curve which will ensure the prescribed temperature–time relation at the active surface of such a thermoelectric battery. The solution of this problem requires essentially answers to two questions. Inasmuch as the rate of cooling by means of thermoelectric devices is limited, one must know first what kind of temperature–time relations can be synthesized by this method. Next, one must find the algorithm for calculating the current–time curve from the given temperature–time relation.

The feasibility of attacking this synthesis problem has been suggested first in [1]. Test data on controlling the surface temperature of thermoelectric cells are given in [2].

We will assume that the temperature distribution $\Theta(\mathrm{Fo}, \chi)$ is described by the following equations:

$$\frac{\partial^2\Theta}{\partial\chi^2} = \frac{\partial\Theta}{\partial(\mathrm{Fo})} - \nu^2(\mathrm{Fo}) \quad (0<\chi<1,\ \mathrm{Fo}>0), \tag{1}$$

$$\left.\frac{\partial\Theta}{\partial\chi}\right|_{\chi=0} - \nu(\mathrm{Fo})\,\Theta^{(0)}(\mathrm{Fo}) - \eta\,\frac{d\Theta^{(0)}(\mathrm{Fo})}{d(\mathrm{Fo})} - \mathrm{Bi}\,(\Theta^{(0)}(\mathrm{Fo}) - \Theta_0) + \xi\nu^2(\mathrm{Fo}) = 0, \tag{2}$$

$$\Theta(\mathrm{Fo},\ 1) = \Theta_1,\ \Theta(0,\ \chi) = -0.5\nu_0^2\chi^2 + A\chi + B\ \ (\Theta^{(0)}(\mathrm{Fo}) \equiv \Theta(\mathrm{Fo},\ 0)). \tag{3}$$

In Eqs. (1)-(3) we use dimensionless variables and parameters, namely: temperature $\Theta = zT$, current density $\nu = (ed/\lambda)j$, time $\mathrm{Fo} = (\lambda t/cd^2)$, distance $\chi = x/d$, heat transfer rate $\mathrm{Bi} = \alpha d/\lambda$, specific heat of the load material $\eta = g/cd$, and contact resistance $\xi = r/\rho d$ (see [3]). This mathematical model takes into account the generation of Joulean heat along the circuit branches of thermoelectric cell $\nu^2(\mathrm{Fo})$ and at the junction contacts $\xi\nu^2(\mathrm{Fo})$, the Pertier effect $\nu(\mathrm{Fo})\,\Theta^{(0)}(\mathrm{Fo})$, the heat transfer from the cooled object surface, and the thermal capacity of commutator bars as well as of the object connected to them. The hot junctions are considered to remain at a constant temperature Θ_1 and the initial temperature distribution is considered to be matching the initial steady-state current ν_0. The constants A and B are easily defined as follows: $B = [\Theta_1 + \mathrm{Bi}\,\Theta_0 + (0.5 + \xi)\,\nu_0^2]\,(\nu_0 + \mathrm{Bi} + 1)^{-1}$, $A = (\nu_0 + \mathrm{Bi})\,B - \mathrm{Bi}\,\Theta_0 - \xi\nu_0^2$.

Applying the Laplace transformation to Eq. (1), with the boundary conditions (2) and (3), we obtain the following functional relation for the transforms ($\bar{\Theta}^{(0)} = L\{\Theta^{(0)}(\mathrm{Fo})\}$):

Scientific-Research Institute of Agrophysics, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 24, No. 5, pp. 902-907, May, 1973. Original article submitted July 17, 1972.

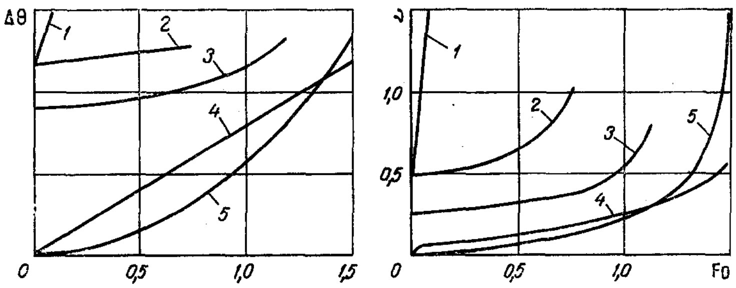

Fig. 1. Synthesis of a monotonic temperature-time relation (Bi = 0, η = 0, ξ = 0.01). Scale on the $\Delta\Theta$axis is 0.05.

$$\overline{v\Theta^{(0)}} + \mathrm{Bi}\,(\overline{\Theta}^{(0)} - \Theta_0 p^{-1}) - \xi\overline{v^2} - Ap^{-1} + \eta\,(p\overline{\Theta}^{(0)} - B)$$
$$= [v_0^2 p^{-1} - \overline{v^2} - (p\overline{\Theta}^{(0)} - \overline{v^2} - B + v_0^2 p^{-1})\,\mathrm{ch}\sqrt{p}\,](\sqrt{p}\,\mathrm{sh}\sqrt{p})^{-1}. \tag{4}$$

It will not be too restrictive to assume that the derivative $\Theta^{(0)}$(Fo) is Laplace transformable along the temperature–time curves $\dot{\Theta}^{(0)}(\mathrm{Fo}) = d\Theta^{(0)}(\mathrm{Fo})/d(\mathrm{Fo})$ to be synthesized at the surface. Then (4) yields the following relation between the controlled temperature $\Theta^{(0)}$(Fo) and the supply current ν(Fo):

$$v(\mathrm{Fo}) = [\Theta^{(0)}(\mathrm{Fo})]^{-1}[\xi v^2(\mathrm{Fo}) + \int_0^{\mathrm{Fo}} K(\mathrm{Fo}-\tau)\,v^2(\tau)\,d\tau + \Phi(\mathrm{Fo},\ \Theta^{(0)}(\mathrm{Fo}))]. \tag{5}$$

Here

$$\Phi(\mathrm{Fo},\ \Theta^{(0)}(\mathrm{Fo})) = -\eta\dot{\Theta}^{(0)}(\mathrm{Fo}) + \mathrm{Bi}\,(\Theta_0 - \Theta^{(0)}(\mathrm{Fo})) + A$$
$$- \int_0^{\mathrm{Fo}} \vartheta_3(1,\ \mathrm{Fo}-\tau)\,\dot{\Theta}^{(0)}(\tau)\,d\tau - v_0^2\int_0^{\mathrm{Fo}} K(\tau)\,d\tau;$$

$$K(\mathrm{Fo}) = \vartheta_3(1,\ \mathrm{Fo}) - \vartheta_0(0,\ \mathrm{Fo});\ \ \vartheta_3(1,\ \mathrm{Fo}) = 1 + 2\sum_{k=1}^{\infty}\exp(-\pi^2k^2\,\mathrm{Fo});$$

$$\vartheta_0(0,\ \mathrm{Fo}) = 1 + 2\sum_{k=1}^{\infty}(-1)^k\exp(-\pi^2k^2\,\mathrm{Fo}).$$

The solution to this nonlinear integral equation (5), which determines the sought control function ν(Fo), can be found for a large class of synthesizable functions $\Theta^{(0)}$(Fo) by the method of successive approximations. In Figs. 1-3 are shown examples of temperature (temperature drop) curves to be synthesized $\Delta\Theta(\mathrm{Fo}) = \Theta_0 - \Theta^{(0)}(\mathrm{Fo})$ and appropriate current curves ν(Fo) calculated from the former according to Eq. (5). The graphs of function $\Delta\Theta$(Fo) in Fig. 1 represent polynomials in time. The temperature drop $\Delta\Theta$(Fo) in Fig. 2 is represented by a piecewise linear function of time, and the use of the appropriate current control ν(Fo) will make it feasible to reduce the time to reach steady-state cooling. A periodic variation of temperature is depicted in Fig. 3, indicating the feasibility, when $\eta \neq 0$, of generating a temperature wave with a jump of $\dot{\Theta}^{(0)}$(Fo) by means of an intermittent current control ν(Fo).

We will now point out some features of Eq. (5) which make feasible its effective solution by the iteration method. Let us consider the synthesis of a monotonic temperature decrease during the time interval [0, E]. Let the temperature–time curve to be synthesized $\Theta^{(0)}$(Fo) belong to the class C^1[0, E] continuous on the interval [0, E] along with its derivative. Then function $\Phi(\mathrm{Fo}) = \Phi(\mathrm{Fo},\ \Theta^{(0)}(\mathrm{Fo}))$ on the right-hand side of Eq. (5) will be positive and continuous. The solution ν(Fo) to Eq. (5) will in this case have to be

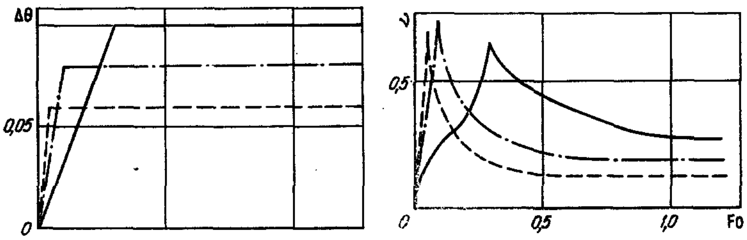

Fig. 2. Accelerated reaching of steady-state temperature (Bi = 0, η = 0, ξ = 0.01).

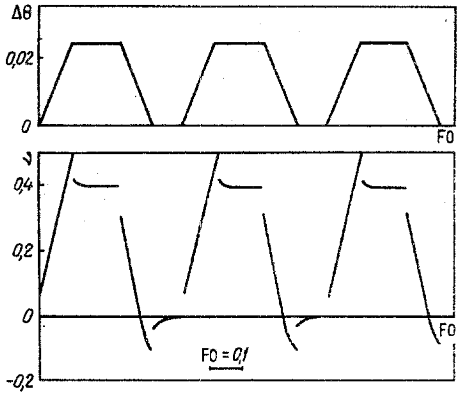

Fig. 3. Synthesis of a trapezoidal periodic temperature variation (Bi = 5, η = 0.1, ξ = 0.01).

sought in the subset D of nonnegative continuous functions in the C[0, E] space. Upon a logical introduction of a partial regularity among elements of set D and upon an examination of Eq. (5), one can ascertain that the right-hand side of this equation is a monotonic continuous operator. By virtue of Schauder's theorem [4], Eq. (5) has on the interval $[0, w_0]$ of set D a solution $\nu(\mathrm{Fo})$ ($\nu \le w_0$) which is unique and which can be obtained by the method of successive approximations, if element w_0 satisfies the condition

$$w_0(\mathrm{Fo})\,\Theta^{(0)}(\mathrm{Fo}) \geqslant \xi w_0^2(\mathrm{Fo}) + \int_0^{\mathrm{Fo}} K(\mathrm{Fo}-\tau)\,w_0^2(\tau)\,d\tau + \Phi(\mathrm{Fo},\ \Theta^{(0)}(\mathrm{Fo})). \quad (6)$$

The sufficient condition (6) for the synthesizability of a monotonic relation $\Theta^{(0)}$ (Fo) lends itself to a simple physical interpretation. If for a given monotonic temperature–time relation $\Theta^{(0)}$ (Fo) there is such a current–time curve w_0(Fo) according to which the Peltier heat output $w_0(\mathrm{Fo})\,\Theta^{(0)}$ (Fo) will at every instant of time exceed the Joule heat input plus the heat input from the ambient medium to the cold junctions Fo $\in$ [0, E], then it is feasible to design a supply current $\nu(\mathrm{Fo}) \le w_0(\mathrm{Fo})$ which will ensure the required cooling characteristic exactly.

We will now apply the method of successive approximations to the determination of the supply current for a thermoelectric battery when $\Theta^{(0)}$ (Fo) is a periodic function. Then, with time Fo increasing, function Φ (Fo) on the right side of Eq. (5) will approach the periodic function Φ^*(Fo) easily calculated from function $\Theta^{(0)}$ (Fo). The steady-state control current for synthesizing the temperature oscillation mode will also be periodic and is described by the equation:

$$\nu(\mathrm{Fo}) = [\Theta^{(0)}(\mathrm{Fo})]^{-1}[\xi\nu^2(\mathrm{Fo}) + \int_{-\infty}^{\mathrm{Fo}} K(\mathrm{Fo}-\tau)\,\nu^2(\tau)\,d\tau + \Phi^*(\mathrm{Fo},\ \Theta^{(0)}(\mathrm{Fo})).$$

Using the method of fast transforms [4], one can indicate the sufficient conditions for both the existence and the uniqueness of solution (7) as well as for the convergence of successive approximations.

Thus, a periodic function $\Theta^{(0)}$ (Fo) which together with its derivative is continuous can be synthesized, if

$$\left|\min_{\mathrm{Fo}} \Theta^{(0)}(\mathrm{Fo}) + 2(1+2\xi)\max_{\mathrm{Fo}} \frac{\Phi^*(\mathrm{Fo})}{\Theta^{(0)}(\mathrm{Fo})}\right| < 2\min_{\mathrm{Fo}} \Theta^{(0)}(\mathrm{Fo}), \quad (8)$$

$$\left|(1+2\xi)\min_{\mathrm{Fo}} \frac{\Phi^*(\mathrm{Fo})}{\Theta^{(0)}(\mathrm{Fo})}\right| < \min_{\mathrm{Fo}} \Theta^{(0)}(\mathrm{Fo}). \quad (9)$$

Inequalities (8) and (9) can be satisfied, if the minimum temperature $\Theta^{(0)}$ (Fo) is bounded from below and if function $|\Phi^*(\mathrm{Fo})|$ and thus also function $|\dot{\Theta}^{(0)}(\mathrm{Fo})|$ are bounded. In the special case of harmonic temperature oscillations with an amplitude a and at a frequency ω, inequalities (8) and (9) are satisfied under the condition that both a and ω are restricted as follows:

$$2a(1+2\xi)[(\eta\omega + R(\omega))^2 + (1+\mathrm{Bi}+S(\omega))^2]^{\frac{1}{2}} < (\Theta_0 - a)^2. \quad (10)$$

Restriction (10) is based on the assumption that $\Theta_0 = \Theta_1$ and that the temperature $\Theta^{(0)}$ (Fo) oscillates symmetrically about Θ_0. Functions $R(\omega)$ and $S(\omega)$ are given in terms of series:

$$R(\omega) = 2\omega\sum_{k=1}^{\infty} \pi^2k^2(\omega^2+\pi^4k^4)^{-1},\quad S(\omega) = 2\omega^2\sum_{k=1}^{\infty}(\omega^2+\pi^4k^4)^{-1}. \quad (11)$$

When $\omega \to 0$, then inequality (10) reduces to the requirement that the amplitude of a generated temperature wave not exceed the maximum steady-state temperature drop. This requirement is the necessary condition for synthesizing low-frequency oscillations.

In conclusion, we note the following. The method of successive approximations, which has been applied here to the synthesis problem, ensures a fast convergence of calculations and is suitable for a large class of functions to be synthesized. At the same time, Eq. (5) can also be successfully solved by other methods as, for example, by methods used in the theory of nonlinear oscillations. These latter methods, when used for the design of thermoelectric batteries operating in the harmonic generator mode, yield a system of algebraic equations from which the Fourier series coefficients for the control function $\nu(\mathrm{Fo})$ can be determined. Such an approach makes it possible to evaluate analytically how the rms current and the current amplitude increase with higher frequencies of the generated wave. It also makes it possible to relate the parameters of the control current to the magnitude of the thermal load.

NOTATION

T	is the absolute temperature;
t	is the time;
x	is the length coordinate;
d	is the height of a thermoelectric cell;
j	is the current density;
Θ_0	is the dimensionless ambient temperature;
λ	is the thermal conductivity of the material of a thermoelectric circuit branch;
c	is the specific heat of the material of a thermoelectric circuit branch, referred to volume;
ρ	is the resistivity of the material of a thermoelectric circuit branch;
e	is the coefficient of thermal emf for a thermocouple;
$z = e^2/\rho\lambda$	is the thermoelectric quality factor;
α	is the coefficient of convective heat transfer;
g	is the thermal capacity of the commutator bars and of the cooled object;
r	is the contact resistance (g and r are both referred to a unit area of active surface in a thermoelectric cell).

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